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ALGEBRAS OVER INFINITE FIELDS, REVISITED

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ABSTRACT

We generalize an earlier result of the first author by showing that under "the usual cardinality-dimension hypothesis" an algebra in which nonzero divisors are invertible is algebraic. This result is then used to show that affine FBN rings over an uncountable algebraically closed field are PI.

Let A be an algebra over the infinite field F. Some years ago, the first author showed [1] that if $x \in A$ and $(x-\lambda)^{-1}$ exists for some $\lambda \in F$, then either x is algebraic over F or the $(x-\lambda)^{-1}$ are linearly independent for those $\lambda \in F$ for which $(x-\lambda)^{-1}$ exists. Among many applications, it was noted that if A is a division algebra and $\operatorname{card} F > \dim_F A$, then A is algebraic over F [2]. Here we offer a wide generalization of this last result:

Theorem 1: *If A is an algebra over the infinite field F, $\operatorname{card} F > \dim_F A$ and the nonzero divisors of A are invertible, then A is algebraic over F.*

This theorem is a consequence of the

Lemma: *If A is an algebra over F and $x \in A$, then the right (left) annihilator ideals, $r(x-\lambda)(\ell(x-\lambda))$, for distinct $\lambda \in F$, form an infinite direct sum of right (left) ideals.*

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The lemma is easily established by observing that the $r(x-\lambda)$ are just the eigenspaces of the $x-\lambda$ considered as linear transformations on the vector space A_F. See, e.g., Lemma 2, [1].

Proof of Theorem 1: Let $x \in A$. By the previous Lemma (applied to both right and left annihilators), $(x-\lambda)^{-1}$ exists for $\operatorname{card} F$ distinct $\lambda \in F$. Therefore, by the remark in the first paragraph, x is algebraic.

We record some applications of Theorem 1. First, let A be an algebra over F with $\operatorname{card} F > \dim_F A$ and M_A a module of finite length. By Fitting's Lemma, we know that regular endomorphisms of M_A are invertible. Now, $\operatorname{End}_A(M_A)$ is a subquotient of some matrix ring over A, so that $\operatorname{End}_A(M_A)$ is also an algebra with $\operatorname{card} F$ exceeding its dimension over F. Thus, by Theorem 1, $\operatorname{End}_A(M_A)$ is algebraic over F.

In another direction, recall that a (right) fully bounded Noetherian (FBN) ring is a right Noetherian ring for which the prime images have the property that essential right ideals contain nontrivial two-sided ideals. The principal examples are, of course, rings satisfying a polynomial identity (PI). We aim to show that the affine (i.e., finitely generated) right FBN rings over an uncountable algebraically closed field are, indeed, PI! For background on Noetherian and PI rings we refer the reader to [3].

THEOREM 2: *If A is a (right) FBN algebra over the algebraically closed field F with $\operatorname{card} F > \dim_F A$, then A satisfies a polynomial identity.*

We break up the proof of this theorem into several reductions:

(1) We can obviously assume that A is not finite dimensional.

(2) If A is not a PI ring, choose a two-sided ideal I maximal with respect to A/I not being a PI ring. It is easy to see that I must be a prime ideal.

(3) Thus, assume that A is prime, (right) FBN and satisfies the dimension hypothesis of the theorem, yet A is not a PI ring, but every proper homomorphic image is PI. Denote by $T_n(A)$ the two-sided ideal generated by all evaluations on A of the identities of the $n \times n$ matrices over F. For each $n = 1, 2, \ldots, T_n(A) \neq (0)$ because A is not a PI ring. However, each nonzero ideal of A contains some $T_n(A)$ by our reduction. (Note: A PI ring satisfies all identities of $n \times n$ matrices over F, for some n, if its nil radical is nilpotent (see [3], Thm. 6.3.16).)

(4) By the boundedness property each essential right ideal of A contains some

$T_n(A)$. Choose c_n to be a regular element in $T_n(A)$ (we can do this because A is a prime Goldie ring). By adjoining the c_n^{-1} to A we obtain the quotient ring of A (every regular element of A has some multiple which is a c_n). Thus, the quotient ring of A satisfies the hypotheses of Theorem 1 and is algebraic over F. But, the quotient ring is of the form $D_n, n \times n$ matrices over a division ring D. D is algebraic over F which is algebraically closed, so $D = F$ and A is PI — a contradiction.

References

[1] A. S. Amitsur, *Algebras over infinite fields,* Proceedings of the American Mathematical Society **7** (1956), 35–38.

[2] A. S. Amitsur, *Countably generated division algebras over nondenumerable fields,* Bulletin of the Research Council of Israel; Section F **7** (1957), 39.

[3] L. Rowen, *Ring Theory*, Academic Press, New York, 1988.